**Copula**

(1)Univariate C.D.F is a Uniform[0,1] ()

Where

(2)Definition : Copula

A Copula is a multivariate CDF whose univariate marginal distribution are all Uniform[0,1].  
Suppose that has a multivariate CDF with the marginal distribution ()  
By (1) ,. Therefore, is a copula

(3)Information contained

Suppose is the CDF of

Therefore,

With this formula ,

We can see that  
(1) The Copula conclude the dependence structure of the variable   
(2) The Marginal Distribution conclude the stand-alone behavior likelihood of

(4)The density of

(5)Invariant Property

Given and strictly increasing function ()

Consider , then has the same Copula with

The Marginal C.D.F

The Copula of ()

(6)Special Copula

(6.1)Independence Copula

(6.2)co-monotonicity Copula (Perfect Positive Dependency)

The co-monotonicity copula is the CDF of where all components of U are equal

For all d-dimensions Copula

(6.3)counter-monotonicity Copula (Perfect Negative Dependency)

The two-dimensional counter-monotonicity copula is defined as the C.D.F of ()

For all 2-dimensions Copula

While For all d-dimensions Copula (d>2)

But is not a copula

(7)Gaussian and t-Copula

When has multivariate normal  
As the Copula only depends on the dependency within , so dependence only on the correlation matrix and it is denoted as   
If any random vector has a Guassian copula , is called to have a meta-Gaussian distribution

Similar to t-destruction

Where is the tail index indicating the tail dependence between the component

(8) Archimedean Copula

An Archimedean Copula with a strict generator has the form :

Where the generator function has the following conditions :

1. *is a continuous, strictly decreasing and convex function mapping [0,1] onto [0,]*

(8.1) Frank Copula

The Frank Copula has generator

The inverse generator is

The bivariate Frank Copula :

(8.2) Clayton Copula

The Clayton Copula has generator function

The Clayton Copula :

We define the Clayton Copula for as

The bivariate Clayton Copula :

(8.3) Gumbel Copula

The Gumbel copula has the generator

The Gumbel Copula

(8.4) Joe Copula

The Joe copula has the generator

The Bivariate Joe Copula

(9) Rank Correlation

(9.1) Rank

For a random variable with sample

The Rank of is defined as follows :

An Important Property : Any increasing function preserve the rank  
CDF also preserve the rank

(9.2) Kendall’s Tau

-concordant pair  
 and are called a concordant pair  
if the ranking of relative to is the same as the ranking of relative to   
Therefore,

-discordant pair  
 and are called a discordant pair if

-Kendall’s tau for

For any increasing function

Based on this property and CDF is an increasing function

Therefore, Kendall’s tau only depends on the Copula of

-Kendall’s tau Correlation Matrix

For a random vector

-Sample Kendall’s tau

The Relation between Copula generation function and Kendall rank correlation tau

Proof:

(1)

Note that

(2)

Where

(3)

By the substitution ,   
Then the Jocobian

Then

(9.3) Spearman’s Rank Correlation Coefficient

Spearman’s correlation coefficient is simply the usual Pearson correlation calculated from the marginal ranks of the data (Which can be inferred from the CDF transformation)

-Spearman’s Rank correlation coefficient

Therefore, Spearman’s Rank correlation coefficient only depends on the Copula of

-Sample Spearman’s Rank correlation coefficient

-Spreaman’s correlation coefficient Matrix

(10)Tail Dependence

-Lower tail dependence

-Upper tail dependence

(11)Fitting Copula

Key Idea:

Since the linkage between Joint PDF and Copula are as follow:

Therefore, given dataset

The Likelihood Function :

The Log-likelihood :

Method (1) : Maximum Likelihood

Finding : by MLE

Method (2) : Pseudo – Maximum Likelihood

Step 1 : Estimating the Marginal Distribution for each components in the random vector   
By MLE or Empirical Method -> To Finding the input of Copula {

Step 2 : Maximize over

\*\*\*  
For Gaussian or t Copula

Gaussian Copula is governed by   
t Copula is governed by

We have two Lemma to estimate

(1)  
(2)

(1) Applied on Gaussian and t  
(2) Applied only on t

Adjustment on t Copula Correlation Matrix

Text, letter

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A screenshot of a computer

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